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Quantum Spinor Structures for Quantum Spaces[1]

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A general theory of quantum spinor structures on quantum spaces is presented within the formalism of quantum principal bundles. Quantum analogs of basic objects of the classical theory are constructed: Laplace and Dirac operators, quantum versions of Clifford and spinor bundles, a Hodge $*$-operator, integration operators. Quantum phenomena are discussed, including an example of the Dirac operator associated to a quantum Hopf fibration.

1. INTRODUCTION

The aim of this study is to present a general theory of spinor structures over quantum spaces, in the spirit of noncommutative differential geometry (Connes, 1994). The framework is the theory of quantum principal bundles (Đurđevich, 1996a, 1997), where quantum groups play the role of the structure groups and quantum spaces play the role of base manifolds.

The formalism presented here could be used for developing a theory of fermions over a quantum space-time, appropriate at ultrasmall distances characterized by the Planck length. Our formalism fulfils various conditions proposed in a general axiomatic framework (Connes, 1994). However, some key conditions of Connes (1994) are not satisfied in our formalism. This includes the spectral asymptotics of the quantum Dirac operator, which in our case, could be very different from the classical behavior. Our constructions are not compatible with the Dixmier trace. Our constructions provide a

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coherent framework for the formulation of quantum elliptic operators (Đurđevich, 2000a) and the study of the index theorems.

The results of this paper include as a special case the spin structures studied in Đurđevich (1995), where quantum spin bundles with classical structure groups were considered and it was assumed that the differential calculus over the structure group is classical.

We shall use the general theory of frame structures on quantum principal bundles (Đurđevich, 1999) in order to develop the idea of a quantum space equipped with a metric. We shall explain how to construct a graded *-algebra $\mathfrak{hor}_P$ representing horizontal forms starting from 'abstract coordinate 1-forms' and a quantum principal bundle P. In accordance with Đurđevich (1999), the space of abstract coordinate 1-forms $\mathbb{V}$ will be defined as the left-invariant part of a bicovariant bimodule Ψ over the structure quantum group G. The group G acts on $\mathbb{V}$ by 'orthogonal transformations'. There exists a canonical braid operator $\tau: \mathbb{V} \otimes \mathbb{V} \to \mathbb{V} \otimes \mathbb{V}$ playing the role of the transposition map (Woronowicz, 1989). We shall introduce abstract Levi-Civita connections. These objects contain the whole geometrical information about quantum frame structures.

This is the most subtle part of the formalism, as it requires that we introduce carefully a number of nontrivial conditions on base space M, structure quantum group G, and the bundle P.

After presenting the basic ideas of Đurđevich (1999), we shall consider special conditions which will further justify interpretation of M as a quantum space equipped with a metric. This includes analytic conditions, the existence of the C^*-algebraic completions of both the base space and the bundle *-algebras, as well as the existence of a 'homogeneous' measure on the base space M. Combining this measure with the integration along the fibers of P, we shall construct a natural measure on the bundle.

One of the main, purely algebraic extra conditions will be the existence of a 'volume element' in the algebra of coordinate horizontal forms. With the help of the volume element and the measure on P, it will be possible to construct the integration map $\int_P: \mathfrak{hor}_P \to \mathbb{C}$.

A quantum version of the Euclidean structure on $\mathbb{V}$ will be represented by a metric form $g: \mathbb{V} \otimes \mathbb{V} \to \Sigma$, where Σ is a *-algebra of *abstract metric tensor coefficients*. In general, $\Sigma \neq \mathbb{C}$, and there exist deep reasons why it is necessary to assume that components of the metric generate a noncommutative *-algebra.

Our definition of a quantum Clifford algebra in Section 3 is motivated by considerations presented in Đurđevich and Oziewicz (1996) and Oziewicz, (1997), based on deformations of braided exterior algebras. Our main condition is similar, the vector space $\mathbb{V}^\wedge$ is equipped with a new product, however, this new product is a quantization of $\wedge$ with a noncommutative deformation parameter.

An extended version of the paper containing all the proofs, which will not be published elsewhere, and also including braided Clifford algebras presented in this issue, is available for download at the author's website.

2. QUANTUM RIEMANNIAN GEOMETRY

We shall recall the definition and basic properties of quantum principal bundles (Đurđevich, 1996a, 1997) and frame structures (Đurđevich, 1999) on them. Quantum frame structures allow us to incorporate into the noncommutative context a fundamental concept of coordinate 1-forms. This level is sufficient to introduce general metric connections together with Levi-Civita connections, covariant derivative, curvature, and torsion operators.

In order to focus on 'true metric spaces', we shall introduce some analytical properties. We shall introduce the integration operators for both the frame bundle and the base, the Laplace operator, the adjoint differential, and the Hodge $*$-operator.

Let G be a compact matrix quantum group (Woronowicz, 1987b), formally represented by a Hopf *-algebra $\mathcal{A}$. We shall denote by $\phi\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ the corresponding coproduct map. We shall use the symbols $\kappa\colon \mathcal{A} \to \mathcal{A}$ and $\epsilon\colon \mathcal{A} \to \mathbb{C}$ for the antipode and the counit map, respectively.

Let M be a compact quantum space represented by a *-algebra $\mathcal{V}$ and let $P = (\mathcal{B}, i, F)$ be a quantum principal G-bundle over M. By definition (Đurđevich, 1997), this means, that $\mathcal{B}$ is a *-algebra, $i\colon \mathcal{V} \to \mathcal{B}$ is a *-monomorphism, and $F\colon \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ is a counital *-homomorphism such that the following properties hold:

i. *The action property.* The following diagram is commutative:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{F} & \mathcal{B} \otimes \mathcal{A} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\phi} \\ \mathcal{B} \otimes \mathcal{A} & \xrightarrow[F\otimes \mathrm{id}]{} & \mathcal{B} \otimes \mathcal{A} \otimes \mathcal{A} \end{array}$$

ii. *The 'orbit space' identification*: $i(\mathcal{V}) = \{b \in \mathcal{B} | F(b) = b \otimes 1\}$.

iii. *The freeness condition*. A linear map $X\colon \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ given by

$$X(q \otimes b) = qF(b)$$

is surjective. If the map X is surjective, then it will be automatically injective, so that X is bijective (Đurđevich, 1996b). Thus, the freeness condition leads into the Hopf–Galois extensions (Schneider, 1994).

In a special case of the frame structures representing 'vanilla' quantum Riemannian manifolds, the above freeness condition will be satisfied automatically.

Let Ψ be a bicovariant (Woronowicz, 1989) bimodule over G. The corresponding left and right co/action maps will be denoted by $l_\Psi\colon \Psi \to \mathscr{A} \otimes \Psi$ and $\wp_\Psi\colon \Psi \to \Psi \otimes \mathscr{A}$, respectively. Let $\mathbb{V} = \Psi_{inv}$ be the corresponding left-invariant part. There exists a natural identification $\Psi \leftrightarrow \mathscr{A} \otimes \mathbb{V}$ of left $\mathscr{A}$-modules. The structure of Ψ is encoded in the restricted right action $\varkappa = (\wp_\Psi|\mathbb{V})\colon \mathbb{V} \to \mathbb{V} \otimes \mathscr{A}$ and a natural right $\mathscr{A}$-module structure $\circ$ on $\mathbb{V}$, given by $\vartheta \circ a = \kappa(a^{(1)})\vartheta a^{(2)}$. If Ψ is *-covariant, then the space $\mathbb{V}$ is *-invariant. We have the following compatibility conditions between $*$, $\circ$, and $\varkappa$:

$$\varkappa * = (* \otimes *)\varkappa, \qquad (\theta \circ a)^* = \theta^* \circ \kappa(a)^*$$

$$\varkappa(\theta \circ a) = \sum_k (\theta_k \circ a^{(2)}) \otimes \kappa(a^{(1)})c_k a^{(3)}, \qquad \sum_k \theta_k \otimes c_k = \varkappa(\theta)$$

We can assume that an auxiliary $\varkappa$-invariant scalar product $(|)$ is defined on $\mathbb{V}$. However, for the purposes of our main considerations, the central role will be played by a *noncommutative scalar product* in $\mathbb{V}$, taking its values in an appropriate *-algebra Σ, generated by 'abstract metric tensor coefficients'.

Let $\tau\colon \mathbb{V} \otimes \mathbb{V} \to \mathbb{V} \otimes \mathbb{V}$ be the canonical braid operator (Woronowicz, 1989) associated to Ψ. It is computed in terms of $\varkappa$ and $\circ$ as

$$\tau(\eta \otimes \vartheta) = \sum_k \vartheta_k \otimes (\eta \circ c_k)$$

Let $\mathbb{V}^\wedge$ be the corresponding τ-exterior algebra obtained from $\mathbb{V}^\otimes$ by factorizing through the space of quadratic relations $\mathrm{im}(I + \tau)$. At this point, it is natural to assume that $\ker(I + \tau) \neq \{0\}$. This ensures the nontriviality of the higher order part of $\mathbb{V}^\wedge$.

In what follows, the algebras $\mathbb{V}^\otimes$ and $\mathbb{V}^\wedge$ will be equipped with the induced $\circ$, $*$, and $\varkappa$-structures (these induced structures will be denoted by the same symbols). The extended structures are constructed by postulating

$$\varkappa(\vartheta\eta) = \varkappa(\vartheta)\varkappa(\eta), \qquad \varkappa(1) = 1 \otimes 1$$

$$(\vartheta\eta)^* = (-)^{\partial\vartheta\partial\eta}\eta^*\vartheta^*$$

$$(\vartheta\eta) \circ a = (\vartheta \circ a^{(1)})(\eta \circ a^{(2)}), \qquad 1 \circ a = \epsilon(a)1$$

The following identities express mutual compatibility between τ and the maps $*$, $\circ$, and $\varkappa$:

$$\begin{array}{ccc} \mathbb{V} \otimes \mathbb{V} & \xrightarrow{\varkappa} & \mathbb{V} \otimes \mathbb{V} \otimes \mathscr{A} \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau\otimes\mathrm{id}} \\ & \varkappa & \end{array}$$

$$\mathbb{V} \otimes \mathbb{V} \longrightarrow \mathbb{V} \otimes \mathbb{V} \otimes \mathcal{A}$$

$$\tau * = * \tau^{-1}, \qquad \tau(\psi \circ a) = \tau(\psi) \circ a$$

Denote by $C_\varkappa$: $\mathbb{V} \to \mathbb{V}$ the canonical intertwiner between $\varkappa$ and its second contragradient $\varkappa^{cc}$. By construction, this map is positive and satisfies

$$\varkappa C_\varkappa = (C_\varkappa \otimes \kappa^2)\varkappa, \qquad \mathrm{tr}(C_\varkappa) = \mathrm{tr}(C_\varkappa^{-1})$$

We shall assume that the scalar product on $\mathbb{V}$ is such that

$$(x^*, y^*) = (y, C_\varkappa, x) \qquad \forall x, y \in \mathbb{V} \tag{1}$$

To put it another way, we can *define* $C_\varkappa$ by the above formula. This implies

$$*C_\varkappa * = C_\varkappa^{-1}, \qquad C_\varkappa = [*]^\dagger *$$

The operator $C_\varkappa$ is associated to the modular properties of the Haar measure (Woronowicz, 1987b). Polary decomposing the map $*$: $\mathbb{V} \to \mathbb{V}$, we obtain

$$* = J_\varkappa C_\varkappa^{1/2} = C_\varkappa^{-1/2} J_\varkappa$$

where $J_\varkappa$: $\mathbb{V} \to \mathbb{V}$ is an antiunitary involution (in other words, $J_\varkappa = J_\varkappa^\dagger = J_\varkappa^{-1}$).

When dealing with various 'coordinate expressions', we shall use a fixed basis $\{\theta_1, \ldots, \theta_d\}$ in $\mathbb{V}$. We shall assume that these vectors satisfy

$$(\theta_i, \theta_j) = \delta_{ij}, \qquad J_\varkappa(\theta_i) = \theta_i$$

In this basis, the representation $\varkappa$: $\mathbb{V} \to \mathbb{V} \otimes \mathcal{A}$ is given by a unitary matrix $[\varkappa_{ij}]$, so that

$$\varkappa(\theta_i) = \sum_j \theta_j \otimes \varkappa_{ji}, \qquad C_\varkappa^{1/2}[\varkappa]C_\varkappa^{-1/2} = [\overline{\varkappa}]$$

The matrix $[C_\varkappa^{1/2}]_{ij} = (\theta_i, C_\varkappa^{1/2}\theta_j)$ is orthogonal.

A 'quantum Euclidean' structure on $\mathbb{V}$ will be specified by a quadratic form g: $\mathbb{V} \otimes \mathbb{V} \to \Sigma$, playing the role of the metric, where Σ is a *-algebra generated by matrix elements of g and g^{-1}, together with a new regular braid operator σ: $\mathbb{V} \otimes \mathbb{V} \to \mathbb{V} \otimes \mathbb{V}$ expressing the twisting properties of Σ and $\mathbb{V}$. This reflects a fundamental property of our theory, the noncommutativity of the metric tensor coefficients and the braided nature of σ. In general, $\sigma \neq \tau$.

The full set of properties involving g, σ and Σ is discussed in Đurđevich (2000b). We shall also assume that $\{\sigma, \tau\}$ form a pair of 'coupled' braid operators, so that the following natural identifications hold for each $n \geq 2$:

$$\{\tau\text{-antisymmetric } n\text{-tensors}\} \leftrightarrow \mathrm{im}(A^n_\sigma) \tag{2}$$

Here A^n_σ: $\mathbb{V}^{\otimes n} \to \mathbb{V}^{\otimes n}$ are the braided τ-antisymmetrizers. By definition, τ-antisymmetric tensors change the sign under all τ-transpositions. Our τ-exterior algebra will be realizable as follows:

$$\mathbb{V}^\wedge \leftrightarrow \mathbb{V}^\otimes/\ker(A_\sigma) \leftrightarrow \mathrm{im}(A_\sigma)$$

These identifications are constructed with the help A_σ. The grading and the *-structure are preserved in this picture. The maps $C_\varkappa$ and $J_\varkappa$ will be extended (by multiplicativity/anti) to $\mathbb{V}^\otimes$. The space $\mathbb{V}^\wedge$ is invariant under the action of these maps. As discussed in Đurđevich (2000b), there is a natural twisted tensor product of algebras $\mathbb{V}^\wedge$ and Σ, and in such a way, we obtain an extended braided exterior algebra $\mathbb{V}^\wedge_\Sigma$. This is a braided exterior algebra built over $\mathbb{V}_\Sigma$ with the help of the extended braid σ: $\mathbb{V}_\Sigma \otimes_\Sigma \mathbb{V}_\Sigma \to \mathbb{V}_\Sigma \otimes_\Sigma \mathbb{V}_\Sigma$.

A *frame structure* on a quantum principal bundle P is given by a graded *-algebra $\mathfrak{hor}_p$ equipped with a first-order Hermitian antiderivation ∇: $\mathfrak{hor}_P \to \mathfrak{hor}_P$ (Đurđevich, 1999). The algebra $\mathfrak{hor}_P$ is defined as $\mathfrak{hor}_P \leftrightarrow \mathcal{B} \otimes \mathcal{V}^\wedge$ at the level of vector spaces, while the product and the *-structure are given by

$$(q \otimes \vartheta)(b \otimes \eta) = \sum_k qb_k \otimes (\vartheta \circ c_k)\eta$$

$$(b \otimes \vartheta)^* = \sum_k b_k^* \otimes (\vartheta^* \circ c_k^*), \qquad \sum_k b_k \otimes c_k = F(b)$$

The elements of $\mathfrak{hor}_P$ are quantum horizontal forms. The elements of $\mathbb{V}^\wedge$, viewed in the framework of $\mathfrak{hor}_P$, are analogs of natural 'coordinate forms' in classical theory of frame bundles. We see that $\mathfrak{hor}_P^0 = \mathcal{B}$. The maps F and $\varkappa$: $\mathbb{V}^\wedge \to \mathbb{V}^\wedge \otimes \mathcal{A}$ naturally combine to a unital *-homomorphism $F^\wedge$: $\mathfrak{hor}_P \to \mathfrak{hor}_P \otimes \mathcal{A}$ satisfying

$$(\mathrm{id} \otimes \phi)F^\wedge = (F^\wedge \otimes \mathrm{id})F^\wedge, \qquad (\mathrm{id} \otimes \epsilon)F^\wedge = \mathrm{id}$$

The map $F^\wedge$ plays the role of the right action of G on horizontal forms. The corresponding. $F^\wedge$-fixed-point graded *-subalgebra $\Omega_M \subseteq \mathfrak{hor}_P$ plays the role of the differential forms on the base manifold M. Accordingly, $\Omega_M^0 = \mathcal{V}$.

The map ∇: $\mathfrak{hor}_P \to \mathfrak{hor}_P$ corresponds to the Levi-Civita connection. By definition, ∇ intertwines the action $F^\wedge$, and ∇ vanishes on the subalgebra generated by $\nabla(\mathcal{V})$ and $\mathbb{V}^\wedge$

It is assumed that there exist linear maps b_α: $\theta \mapsto b_\alpha(\theta) \in \mathcal{B}$ and elements $f_\alpha \in \mathcal{V}$ satisfying a completeness condition,

$$1 \otimes \theta = \sum_\alpha b_\alpha(\theta)\, \nabla(f_\alpha) \tag{3}$$

$$F[b_\alpha(\theta)] = (b_\alpha \otimes \mathrm{id})\varkappa(\theta) \tag{4}$$

From the above-mentioned postulates, it follows that ∇ is reduced in Ω_M and that the restriction map ${}^M d\colon \Omega_M \to \Omega_M$ is a Hermitian differential (corresponding to the standard exterior derivative of differential forms). It can be shown that Ω_M is generated by $\mathcal{V}$ and ${}^M d(\mathcal{V})$.

We can introduce a 'coordinate' description of ∇,

$$\nabla(b) = \sum_i \partial_i(b) \otimes \theta_i, \qquad b \in \mathcal{B} \tag{5}$$

The maps $\partial_i\colon \mathcal{B} \to \mathcal{B}$ are counterparts of canonical horizontal coordinate vector fields. They completely determine ∇. The $F^\wedge$ covariance of ∇ is equivalent to the property

$$F\partial_i(b) = \sum_{jk} \partial_j(b_k) \otimes c_k \kappa^{-1}(\varkappa_{ij})$$

$$F(b) = \sum_k b_k \otimes c_k, \qquad \sum_j \theta_j \otimes \varkappa_{ji} = \varkappa(\theta_i)$$

Let $\mu\colon \mathcal{B} \to M_d(\mathcal{B})$ be a *-homomorphism defining the right $\mathcal{B}$-module structure on $\mathfrak{hor}_P$,

$$\theta_i b = \sum_j \mu_{ij}(b)\theta_j$$

The graded Leibniz rule for ∇ is translated into the system of equations

$$\partial_i(qb) = q\partial_i(b) + \sum_j \partial_j(q)\mu_{ji}(b)$$

Let $\nu\colon \mathcal{A} \to M_d(\mathbb{C})$ be a unital homomorphism given by $\theta_i \circ a = \Sigma_j\, \nu_{ij}(a)\theta_j$. The maps $\mu_{ij}\colon \mathcal{B} \to \mathcal{B}$ are expressible via the right $\mathcal{A}$-module structure on $\mathbb{V}$ and the map F as

$$\mu_{ij}(b) = \sum_k b_k \nu_{ij}(c_k)$$

The introduced coordinate vector fields fit into a general framework for quantum vector fields introduced in Borowiec (1996; see also Borowiec, 1997; Borowiec *et al.*, 2000).

The following identity expresses the compatibility between $\circ$ and the $*$-structure on $\mathbb{V}$:

$$\nu[\kappa(a)^*] = C_\varkappa^{-1/2}\overline{\nu(a)}C_\varkappa^{1/2} \tag{6}$$

The Hermiticity property of ∇ in terms of the maps ∂_i is

$$\partial_i = \sum_j [C_\varkappa^{1/2}\mu]_{ji}\{\partial_j^*\}, \qquad \partial_j^* = {}^*\partial_i{}^* \tag{7}$$

The structure group G corresponds to a transformation group of 'local orthonormal frames'. The frame structure allows us to think of P as the bundle of ‘orthonormal frames’ over M. In accordance with this analogy, it would be natural to assume that $\varkappa$ is faithful (in other words, G is completely determined by its action on $\mathbb{V}$, which means that $\mathcal{A}$ is generated by the matrix elements of $\varkappa$). However, from the point of view of our spinorial constructions, it is natural to allow the situations where $\varkappa$ is not faithful ($G \leftrightarrow \mathrm{Spin}(m)$). This allows us to include the spin structures within the framework of the frame structures.

Lemma 2.1. In the case of ‘trully frame bundles’, when $\varkappa$ is faithful, the freeness condition for F is automatically fulfilled.

We are now going to write down commutation relations between coordinate vector fields ∂_i, involving the curvature tensor. Let us recall that the curvature $\rho_\nabla\colon \mathcal{A} \to \mathfrak{z}\mathfrak{h}^2_P$ of ∇ is uniquely determined through a fundamental identity

$$\nabla^2(b) = -\sum_k b_k \rho_\nabla(c_k) \tag{8}$$

Here $\mathfrak{z}\mathfrak{h}_P$ is the graded commutant of Ω_M in $\mathfrak{hor}_P$. The fact that the curvature always take values from $\mathfrak{z}\mathfrak{h}^2_P$ implies strong constraints for possible forms of a Levi-Civita connection in noncommutative geometry.

The canonical inclusion of the exterior algebra into the tensor algebra allows us to define the ‘components’ $\rho^{ij}_\nabla\colon \mathcal{A} \to \mathcal{B}$ of the curvature by the formula

$$\rho_\nabla(a) = \frac{1}{2}\sum_{ij} \rho^{ij}_\nabla(a) \otimes \{\theta_i \otimes \theta_j\}$$

The components by construction satisfy τ-antisymmetricity relations

$$\rho^{ij}_\nabla = -\sum_{kl} \tau^{ij}_{kl}\rho^{kl}_\nabla, \qquad \tau(\theta_i \otimes \theta_j) = \sum_{kl} \tau^{kl}_{ij}\theta_k \otimes \theta_l$$

Lemma 2.2.

$$\partial_i\partial_j(b) - \sum_{kl} \sigma^{ij}_{kl}\, \partial_k\partial_l(b) + \tfrac{1}{2}\sum_\alpha b_\alpha \rho^{ij}_\nabla(c_\alpha) = 0$$

$$\sum_{kl} \sigma^{kl}_{ij}\theta_k \otimes \theta_i = \sigma(\theta_i \otimes \theta_j)$$

2.1. Integration Operators

We shall assume that $\mathcal{V}$ is realized as an everywhere dense *-subalgebra of a unital $\mathbb{C}$*-algebra $\hat{\mathcal{V}}$. We shall also assume that a faithful state $\omega_M\colon \hat{\mathcal{V}} \to \mathbb{C}$ is given (representing a ‘measure’ on quantum base space M). By definition, the faithfulness property means that ω_M is *strictly positive* on the positive elements. We shall assume that ω_M admits a modular operator $\Theta: \mathcal{V} \to \mathcal{V}$ of the form

$$\omega_M(fg) = \omega_M(\Theta(g)f) \qquad \forall f, g \in \mathcal{V}$$

The modular operator Θ is uniquely determined by the state ω_M. The following identities hold:

$$\Theta(fg) = \Theta(f)\Theta(g), \qquad \Theta * \Theta = *$$

and we see that Θ is necessarily bijective.

We shall introduce compatibility conditions between the quantum principal bundle P with the frame structure and the measure ω_M with the associated modular automorphism Θ.

Let us introduce a 'vertical integration' operator $\int_\uparrow: \mathcal{B} \to \mathcal{V}$ by

$$i\left[\int_\uparrow (b)\right] = (\mathrm{id} \otimes h)F(b)$$

where $h: \mathcal{A} \to \mathbb{C}$ is the Haar measure (Woronowicz, 1987b) of G.

First Assumption on P—Strict Positivity. The map $\int_\uparrow$ is strictly positive,

$$\forall b \in \mathcal{B}, \quad \int_\uparrow (b^*b) \geq 0, \quad \text{and if} \quad \int_\uparrow (b^*b) = 0, \quad \text{then} \quad b = 0$$

This property ensures that the *-algebra $\mathcal{B}$ is closable into a $\mathbb{C}$*-algebra $\hat{\mathcal{B}}$, using a natural GNS-type faithful *-representation by bounded operators.

By combining $\int_\uparrow$ and ω_M, we arrive at a faithful state $\omega_P: \hat{\mathcal{B}} \to \mathbb{C}$ playing the role of the 'measure' on the bundle P. Let us assume that ω_P admits a modular operator $\hat{\Theta}: \mathcal{B} \to \mathcal{B}$.

Lemma 2.3. We have $F\hat{\Theta} = (\hat{\Theta} \otimes \kappa^2)F$, $\hat{\Theta}[\mathcal{V}] = \mathcal{V}$, and $\hat{\Theta} \restriction \mathcal{V} = \Theta$.

The space $\mathcal{B}$ is equipped with a natural F-invariant scalar product given by

$$\langle b, q\rangle = \omega_P(b^*q)$$

Playing with the definitions of $\hat{\Theta}$ and $\langle\ ,\ \rangle$, it follows that $*: \mathcal{B} \to \mathcal{B}$ is formally adjointable. Accordingly,

$$\langle b, q^*\rangle = \langle q, [*]^\dagger(b)\rangle \quad \forall q, b \in \mathcal{B}, \qquad [*]^\dagger = \hat{\Theta}^{-1} * = *\hat{\Theta}$$

Let $\mathcal{T}$ be a complete set of mutually nonequivalent irreducible representations of G. By decomposing $\mathcal{B}$ into the multiple irreducible submodules relative to the action F, we arrive at

$$\mathcal{B} = \sum_{\alpha\in\mathcal{T}}^{\oplus} \mathcal{B}^\alpha, \qquad \mathcal{B}^\alpha \leftrightarrow \mathcal{E}_\alpha \otimes H_\alpha$$

Here we have used intertwiner bimodules $\mathcal{E}_\alpha = \mathrm{Mor}(\alpha, F)$, and H_α are the corresponding representation spaces. If Ø is the trivial representation in $\mathbb{C}$, then $\mathcal{B}^\alpha \leftrightarrow \mathcal{V}$. For each $\alpha \in \mathcal{T}$, let us denote by $\{\}_\alpha: \mathcal{B} \to \mathcal{B}^\alpha$ the corresponding projection map. The vertical integration map is given by projecting $\mathcal{B}$ onto $\mathcal{V}$, in other words, $\int_\tau \leftrightarrow \{\ \}_{Ø}$.

Second Extra Assumption on P—Horizontal Homogeneity. Consider the elements $b_{\alpha i} = b_\alpha(\theta_i)$. Then

$$\omega_M\left\{\sum_{ij} [C_\varkappa^{-1}]_{ji}\ \partial_i(b_{\alpha j} f)\right\} = 0 \qquad \forall f \in \mathcal{V}$$

This condition expresses the idea that the measure on M is 'homogeneous'.

The maps $\partial_i: \mathcal{B} \to \mathcal{B}$ play the role of canonical horizontal vector fields, and ω_M should be invariant under the appropriate 'infinitesimal horizontal transformations'. The first-order 'differential' operators $T_\alpha: \mathcal{V} \to \mathcal{V}$ defined by

$$T_\alpha(f) = \sum_{ij} [C_\varkappa^{-1}]_{ji}\partial_i(b_{\alpha j} f)$$

figure in the above expression. These operators are naturally associated to the frame structure.

Lemma 2.4. Under the above homogeneity and positivity assumptions, we have

$$\omega_P[\partial_i(b)] = 0 \qquad \forall b \in \mathcal{B}$$

It is interesting to calculate the formally adjoint operators for important coordinate maps $\mu_{ij}: \mathcal{B} \to \mathcal{B}$ and $\partial_i: \mathcal{B} \to \mathcal{B}$.

Lemma-2.5. The maps μ_{ij} and ∂_i are formally adjointable and

$$\mu^\dagger(b) = C_\varkappa^{1/2}\mu(b)C_\varkappa^{-1/2} \tag{9}$$

$$-\partial_i^\dagger = \sum_j [C_\varkappa^{-1/2}]_{ji}\partial_j = \sum_j \mu_{ji}^\dagger\{\partial_j^*\} \tag{10}$$

We are going to construct the integration operator for horizontal forms. This will be done by combining the measure ω_P and a 'coordinate volume form' the existence of which will be ensured by our next extra condition.

Third Extra Condition—Self-Duality of Coordinate Forms. There exists a number $m \in \mathbb{N}$ such that $\mathbb{V}^{\wedge m} \leftrightarrow \mathbb{C}$ and $\mathbb{V}^{\wedge k} = \{0\}$ for each $k > m$.

We can introduce the 'volume element' as a single generator $w = w^* \in \mathbb{V}^{\wedge m}$. The above condition follows from the simple assumption that the braided exterior algebra $\mathbb{V}^{\wedge}$ is finite dimensional. Let us observe that the following symmetry property holds:

$$\mathbb{V}^{\wedge k} \leftrightarrow \mathbb{V}^{\wedge m-k}$$

This is because the formula

$$j(x, y)w = x \wedge y$$

defines a nondegenerate pairing $j\colon \mathbb{V}^{\wedge k} \times \mathbb{V}^{\wedge m-k} \to \mathbb{C}$. It follows that there exists a unique grade-preserving map $\diamondsuit\colon \mathbb{V}^{\wedge} \to \mathbb{V}^{\wedge}$ such that

$$j(y, x) = (-)^{\partial x \partial y} j(\diamondsuit(x), y) \qquad \forall x, y \in \mathbb{V}^{\wedge}$$

From the definition of w, it follows that

$$\varkappa(w) = w \otimes Q \qquad w \circ a = \lambda(a)w$$

where $Q \in \mathcal{A}$ is a 'quantum determinant' such that

$$\phi(Q) = Q \otimes Q, \qquad \kappa(Q) = Q^{-1} = Q = Q^*$$

and $\lambda\colon \mathcal{A} \to \mathbb{C}$ is a unital multiplicative functional satisfying

$$\overline{\lambda(a)} = \lambda[\kappa(a)^*]$$

In general, $m \neq d$, in contrast with classical geometry.

We shall introduce the integration map $\int_P\colon \mathfrak{hor}_P \to \mathbb{C}$,

$$\int_P (b \otimes \vartheta) = \begin{cases} \omega_P(b) & \text{for} \quad \vartheta = w \\ 0 & \text{if} \quad \deg \vartheta < m \end{cases}$$

Lemma 2.6. $\int_P \nabla\varphi = 0$, $\forall \varphi \in \mathfrak{hor}_P$, and

$$\int_P [\varphi^*] = \left(\int_P \varphi\right)^*, \qquad \sum_k \left(\int_P \varphi_k\right) \otimes c_k = \left(\int_P \varphi\right) \otimes Q,$$

$$\sum_k \varphi_k \otimes c_k = F^{\wedge}(\varphi)$$

A Hermitian involution Q naturally decomposes into Hermitian projections. Q_+, Q_- where $Q_\pm = 1/2 \pm Q/2$. If the map $\lambda\colon \mathcal{A} \to \mathbb{C}$ is in addition *central,* then it will be possible to pass to the corresponding 'components' of G, determined by $Q_\pm$. In particular, we can factorize through the Hopf *-ideal generated by Q_-, reducing to the case $Q = 1$. This is the quantum version of *unimodularity,* passing from $O(d)$ to $SO(d)$ groups.

Geometrically, such a restriction means that we are dealing with *oriented manifolds*. In what follows, it will be assumed that orientability property holds (and centrality of λ, as a necessary consistency condition).

Lemma 2.7. $\int_P \varphi\psi = (—)^{\partial\varphi\partial\psi} \int_P \Lambda(\psi)\varphi$, where $\Lambda\colon \mathfrak{hor}_P \to \mathfrak{hor}_P$ is a grade-preserving homomorphism defined by $\Lambda(b \otimes \vartheta) = \hat{\Theta}(b_\lambda) \otimes \diamondsuit(\vartheta)$ and $[\ ]_\lambda = (\mathrm{id} \otimes \lambda)F$.

We shall make extensive use of an *extended horizontal forms* algebra $\mathfrak{hor}_{P,\Sigma}$ obtained by mixing Σ with the standard horizontal forms. More precisely, $\mathfrak{hor}_{P,\Sigma}$ is obtained by taking the twisted tensor product between $\mathcal{B}$ and the extended braided exterior algebra $\mathbb{V}^\wedge_\Sigma$. We have obviously natural left/right $\mathcal{B}$, Σ-module identifications

$$\mathfrak{hor}_{P,\Sigma} \leftrightarrow \Sigma \otimes \mathfrak{hor}_P \leftrightarrow \mathfrak{hor}_P \otimes \Sigma$$

The action $F^\wedge$ naturally extends, with the help of $\varkappa_\Sigma\colon \Sigma \to \Sigma \otimes \mathcal{A}$, to the action $F^\wedge\colon \mathfrak{hor}_{P,\Sigma} \to \mathfrak{hor}_{P,\Sigma} \otimes \mathcal{A}$. The frame structure ∇ naturally extends, by Σ-left/right linearity, to a Hermitian antiderivation $\nabla\colon \mathfrak{hor}_{P,\Sigma} \to \mathfrak{hor}_{P,\Sigma}$

The base space algebra Ω_M is naturally included in $\Omega_{M,\Sigma}$, which is defined as the $F^\wedge$-fixed point subalgebra of $\mathfrak{hor}_{P,\Sigma}$. We shall freely pass from extended to nonextended objects and vice versa.

2.2. The Hodge Operator

We shall introduce the Hodge $*$-operator on $\mathbb{V}^\wedge_\Sigma$. Then we will extend it to $\mathfrak{hor}_{P,\Sigma}$. We shall assume that $\mathbb{V}^\wedge_\Sigma$ is equipped with a Σ-valued quadratic form $g_\wedge$ and the associated scalar product $\langle\ \rangle$, as explained in Đurđevich (2000b). Let an automorphism $S\colon \Sigma \to \Sigma$ be such that

$$S(\alpha) \circ a = S(\alpha \circ a), \qquad *S* = S^{-1}, \qquad \varkappa_\Sigma S = (S \otimes \mathrm{id})\varkappa_\Sigma$$
$$\mathbb{V}^{\wedge m}_\Sigma = \Sigma w = w\Sigma, \qquad w\alpha = S(\alpha)w$$

The map j is extendible to a Σ-valued pairing acting within $\mathbb{V}^\wedge_\Sigma$,

$$j(\varphi, \psi q) = j(\varphi, \psi)S(q), \qquad j(q\varphi, \psi) = qj(\varphi, \psi), \qquad j(\varphi, q\psi) = j(\varphi q, \psi)$$
$$j(\varphi \circ a^{(1)}, \psi \circ a^{(2)}) = \lambda(a^{(1)})j(\varphi, \psi) \circ a^{(2)}$$

Proposition 2.8. (i) The formula $g_\wedge(x, y) = j(x, \star[y])$ uniquely defines a linear operator $\star\colon \mathbb{V}^\wedge_\Sigma \to \mathbb{V}^\wedge_\Sigma$ such that $\star(\mathbb{V}^{\wedge k}_\Sigma) \subseteq \mathbb{V}^{\wedge m-k}_\Sigma$. (ii) The map $\star$ is bijective, and

$$\star[xq] = \star[x]S^{-1}(q), \qquad \star[qx] = q\star[x], \qquad q \in \Sigma$$
$$\varkappa\star = (\star \otimes \mathrm{id})\varkappa \tag{11}$$

$$\star(\vartheta \circ a_\lambda) = \star(\vartheta) \circ a \tag{12}$$

The automorphism S: $\Sigma \to \Sigma$ has a simple structure, as it is sufficient to calculate its action on the elements of the form $g(x, y)$ where $x, y \in \mathbb{V}$.

We have σ: $w \otimes x \mapsto T(x) \otimes w$ and σ: $y \otimes w \mapsto w \otimes T^*(y)$, where T: $\mathbb{V} \to \mathbb{V}$ is a bijective linear operator. Using this, and iteratively applying the definition of the Σ-bimodule structure on $\mathbb{V}_\Sigma$, we find

$$S\{\langle x, y\rangle\} = \langle T^{-1}(x), T(y)\rangle$$

The map T extends to a unital automorphism T: $\mathbb{V}^\wedge \to \mathbb{V}^\wedge$, and the above formula remains valid for arbitrary elements of braided exterior algebra.

Lemma 2.9. Let the concept of the adjoint operator be appropriately S-twisted, a necessary consistency condition, having in mind a right S-twisted Σ-linearity of $\star$ and T. Then

$$\star^\dagger = \star, \qquad T^\dagger = T, \qquad \langle x, T(y)\rangle = S\{\langle T(x), y\rangle\},$$
$$\langle x, \star(y)\rangle = S^{-1}\{\langle \star(x), y\rangle\}$$

Let $\psi \in \mathbb{V}_\Sigma^{\wedge n}$ be realized in the tensor algebra. The contraction operators $\iota[x]$: $\mathbb{V}_\Sigma^\wedge \to \mathbb{V}_\Sigma^\wedge$ are $\iota[x]\psi = (g \otimes \mathrm{id}^{n-1})(x \otimes \psi)$. These operators are σ-braided antiderivations; they satisfy the σ-braided Leibniz rule (cf. Oziewicz *et al.*, 1995),

$$\iota[x]y + \sum_\alpha y_\alpha\, \iota[x_\alpha] = g(x, y) \qquad \forall x, y \in \mathbb{V}_\Sigma, \qquad \sum_\alpha y_\alpha \otimes x_\alpha = \sigma(x \otimes y)$$

Using a natural Σ-valued scalar product in $\mathbb{V}_\Sigma^\wedge$, it follows that

$$[x \wedge (\ )]^\dagger = \iota[x^*] \qquad \forall x \in \mathbb{V}_\Sigma$$

The contraction operators are the adjoint maps of the multiplication operators.

Lemma 2.10. $\iota[e] = \star^{-1}[e \wedge (\ )]\star$. In other words, $\star$ acts as as a conjugation between multiplication and contraction maps.

2.3. Extension to Horizontal Forms

The operator $\star$ will be extended to $\mathfrak{hor}_{P,\Sigma}$ by left $\mathcal{B}$-linearity,

$$\star_P(b \otimes \vartheta) = b \otimes \star(\vartheta) \qquad \forall b \in \mathcal{B}, \quad \forall \vartheta \in \mathbb{V}_\Sigma^\wedge$$

Here it is necessary to deal with extended horizontal forms $\mathfrak{hor}_{P,\Sigma}$. As a consequence of (11), we have

$$F^\wedge \star_P = (\star_P \otimes \mathrm{id})F^\wedge \tag{13}$$

This intertwining property, together with (12) and the definition of the product in $\mathfrak{hor}_{P,\Sigma}$, implies that $\star_P$ is λ-twisted right $\mathcal{B}$-linear, $\star_P(\psi b_\lambda) = \star_P(\psi)b$.

Definition 2.11. The map $\star_P$ is called the Hodge $*$-operator for P.

Observe that $\star_P(\Omega_{M,\Sigma}) = \Omega_{M,\Sigma}$, as follows from (13). We shall denote by $\star_M\colon \Omega_{M,\Sigma} \to \Omega_{M,\Sigma}$ the restricted map.

The introduced integration map $\int_P\colon \mathfrak{hor}_P \to \mathbb{C}$ naturally extends, by left Σ-linearity, to $\int_P\colon \mathfrak{hor}_{P,\Sigma} \to \Sigma$. Such an extended map intertwines the actions of G and satisfies

$$\int_P [\varphi^*] = S\left\{\int_P [\varphi]^*\right\}, \qquad \int_P [\varphi q] = \int_P [\varphi] S(q) \quad q \in \Sigma$$

Lemma 2.12. The formula $\langle\varphi, \psi\rangle = \int_P \varphi^* \star_P [\psi]$ defines a Σ-valued scalar product in $\mathfrak{hor}_{P,\Sigma}$. This scalar product is G-covariant, and in terms of the natural left $\mathcal{B}$-module identification $\mathfrak{hor}_{P,\Sigma} \leftrightarrow \mathcal{B} \otimes \mathbb{V}^{\wedge}_{\Sigma}$, it is given by a direct product of natural scalar products in $\mathbb{V}^{\wedge}_{\Sigma}$ and $\mathcal{B}$.

In what follows, we shall assume that $\mathfrak{hor}_{P,\Sigma}$ is equipped with the constructed scalar product. The next lemma gives an explicit description of the (formal) adjoint covariant derivative map.

Lemma 2.13. The map $\nabla\colon \mathfrak{hor}_{P,\Sigma} \to \mathfrak{hor}_{P,\Sigma}$ is adjointable; there exists a (necessarily unique) linear map $\nabla^\dagger\colon \mathfrak{hor}_{P,\Sigma} \to \mathfrak{hor}_{P,\Sigma}$ such that

$$\langle\varphi, \nabla(\psi)\rangle = \langle\nabla^\dagger(\varphi), \psi\rangle \qquad \forall\varphi, \psi \in \mathfrak{hor}_P \tag{14}$$

$$\nabla^\dagger(\psi) = -\star_P^{-1}\nabla\star_P(\psi) \tag{15}$$

The adjoint derivative $\nabla^\dagger$ is Σ-bilinear. It intertwines the right action $F^\wedge$,

$$F^\wedge\nabla^\dagger = (\nabla^\dagger \otimes \mathrm{id})F^\wedge, \qquad \nabla^\dagger = -\sum_{i=1}^{d} \partial_i \otimes \iota[\theta_i] \tag{16}$$

This can be proved in several ways; for example, it follows by taking the adjoints of ∂_i and θ_i in the coordinate expression for ∇. Let us observe that $\nabla^\dagger$ generically takes the values from the extended horizontal algebra $\mathfrak{hor}_{P,\Sigma}$.

2.4. Quantum Laplacian

Definition 2.14. A linear operator $\Delta_P = \nabla\nabla^\dagger + \nabla^\dagger\nabla\colon \mathfrak{hor}_{P,\Sigma} \to \mathfrak{hor}_{P,\Sigma}$ is called the quantum Laplacian.

A quantum Laplacian Δ_P is a symmetric positive operator; it operates within the extended horizontal forms algebra.

Proposition 2.15. Let $F(b) = \Sigma_\alpha\, b_\alpha \otimes c_\alpha$ and $g_{ij} = g(\theta_i, \theta_j)$. Then

$$\Delta_P(b \otimes \vartheta) = -\sum_{ij} \partial_i \partial_j(b) \otimes g_{ij}\, \vartheta + \frac{1}{2}\sum_{ij\alpha} b_\alpha \rho_\nabla^{ij}(c_\alpha) \otimes \theta_i \iota[\theta_j](\vartheta)$$

In general, maps g_{ij}: $\mathbb{V}^\wedge_\Sigma \to \mathbb{V}^\wedge_\Sigma$ will be nonscalar operators. Further elementary algebraic properties of Δ_P include the covariance property,

$$F^\wedge \Delta_P = (\Delta_P \otimes \mathrm{id})F^\wedge$$
$$-\Delta_P \star_P^{-1} = \nabla \star_P \nabla + \star_P^{-1} \nabla \star_P \nabla \star_P^{-1}$$
$$-\star_P \Delta_P = \nabla \star_P^{-1} \nabla + \star_P \nabla \star_P^{-1} \nabla \star_P$$

According to (2.4), the map Δ_P is reduced in Ω_M. We shall denote the corresponding restriction map by Δ_M: $\Omega_M \to \Omega_M$,

$$\Delta_M\, {}^M d = {}^M d\, \Delta_M, \qquad \Delta_M\, {}^M d^\dagger = {}^M d^\dagger\, \Delta_M, \qquad \Delta_M = ({}^M d + {}^M d^\dagger)^2$$

3. QUANTUM SPIN BUNDLES

The quantum spinor structure will be defined as ‘covering bundles’ of the ‘true orthonormal frame bundles’. Their structure group will be a kind of a quantum spin group. We shall start by introducing quantum versions of the Clifford bundle algebra and the associated spinor bundle. Here we shall use a theory of braided Clifford algebras (Đurđevich, 2000b). Our definition of a quantum Clifford algebra is motivated by the braided approach developed in Đurđevich and Oziewicz (1996) and Oziewicz (1997), where Clifford algebras are understood as deformations of the braided exterior algebras.

Throughout this section, we shall assume that the structure group G possesses a special ‘spinorial’ representation and consequently we shall relax the faithfulness assumption for $\varkappa$. The frame structures on quantum spaces/bundles are then interpretable as ‘covering bundles’ of the ‘real’ orthonormal frame bundles. The orthogonal quantum group G_0 corresponds to the Hopf *-subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ generated by the matrix dements $\varkappa_{ij}$.

The original orthonormal frame bundle P_0 is given by the *-subalgebra $\mathcal{B}_0$ of $\mathcal{B}$ generated by multiple irreducible submodules of $\mathcal{B}$ corresponding to the representations of G_0. Obviously $F(\mathcal{B}_0) \subseteq \mathcal{B}_0 \otimes \mathcal{A}_0$ and $i(\mathcal{V}) \subseteq \mathcal{B}_0$. Taking the corresponding restriction maps, we obtain a quantum principal G_0-bundle $P_0 = (\mathcal{B}_0, i, F)$ over M with the faithful action $\varkappa$. Geometrically, P is a kind of a covering space for P_0 and P_0 corresponds to the ‘vanilla’ orthormal frame bundle in classical geometry.

We shall formalize the idea of a ‘quantum spinor space’. We shall use the quantum Clifford algebra cl[$\mathbb{V}$, g, σ, Σ] associated to $\mathbb{V}$, metric coefficients algebra Σ, the braid operator σ, and quantum metric g: $\mathbb{V} \otimes \mathbb{V} \to \Sigma$. This algebra is constructed by g-deforming the product in $\mathbb{V}^\wedge_\Sigma$ while preserving the *-structure and the $\circ$-structure, as explained in Đurđevich (2000b).

Let us assume that a finite-dimensional Hilbert space $\mathbb{S}$ is given, together with a unitary representation $\varkappa_{\mathbb{S}}\colon \mathbb{S} \to \mathbb{S} \otimes \mathcal{A}$. Let us also assume that $\mathbb{S}$ is an irreducible left *-module over $\mathrm{cl}[\mathbb{V}, g, \sigma, \Sigma]$. Finally, let us assume that the following compatibility condition holds:

$$\varkappa_{\mathbb{S}}(Z\xi) = \varkappa_{\Sigma}(Z)\varkappa_{\mathbb{S}}(\xi), \qquad Z \in \mathrm{cl}[\mathbb{V}, g, \sigma, \Sigma], \qquad \xi \in \mathbb{S}$$

The meaning of this condition is that the action of G on $\mathrm{cl}[\mathbb{V}, g, \sigma, \Sigma]$ can be viewed as the adjoint action of $\varkappa_{\mathbb{S}}$ in terms of operators acting in $\mathbb{S}$.

Definition 3.1. If the above conditions are fulfilled, we shall say that $\mathbb{S}$ is a quantum spinor space associated to G and $\mathrm{cl}[\mathbb{V}, g, \sigma, \Sigma]$.

The $\mathrm{cl}[\mathbb{V}, g, \sigma, \Sigma]$-module structure $\gamma\colon \mathrm{cl}[\mathbb{V}, g, \sigma, \Sigma] \to B(\mathbb{S})$ is generally not faithful, and Σ may be infinite dimensional. The map γ (including its values on Σ) is completely determined by the assignment $\gamma\colon \mathbb{V} \to B(\mathbb{S})$. This simple observation can be used as a starting point in constructing Σ and $\mathrm{cl}[\mathbb{V}, g, \sigma, \Sigma]$.

By combining the *-algebra structures on $\mathcal{B}$ and $\mathbb{V}^{\wedge}_{\Sigma} \leftrightarrow \mathrm{cl}[\mathbb{V}, g, \sigma, \Sigma]$, we obtain a *-algebra $\mathfrak{cl}[P]$. By construction, we have a natural identification $\mathfrak{cl}[P] \leftrightarrow \mathfrak{hor}_{P,\Sigma}$ of $\mathcal{B}$-bimodules. The *-algebra structure on $\mathfrak{cl}[P]$ is given by the standard cross-product type formulas

$$(q \otimes \vartheta)(b \otimes \eta) = \sum_k qb_k \otimes (\vartheta \circ c_k)\eta$$

$$(b \otimes \vartheta)^* = \sum_k b_k^* \otimes (\vartheta^* \circ c_k^*)$$

By taking the product of the actions $\varkappa_*$ and F, we obtain a *-homomorphism $F_{\mathrm{cliff}}\colon \mathfrak{cl}[P] \to \mathfrak{cl}[P] \otimes \mathcal{A}$. Obviously, $F_{\mathrm{cliff}} \leftrightarrow F^{\wedge}$, in terms of the identification

$$\mathfrak{cl}[P] \leftrightarrow \mathfrak{hor}_{P,\Sigma}$$

We shall denote by $\mathfrak{cl}[M] \subseteq \mathfrak{cl}[P]$ the F_{cliff}-invariant *-subalgebra of $\mathfrak{cl}[P]$. Obviously, we have a natural identification

$$\mathfrak{cl}[M] \leftrightarrow \Omega_{M,\Sigma}$$

Starting from a quantum spinor space, we can define the associated spinor bundle. Consider a free left $\mathcal{B}$-bimodule $\mathcal{S} = \mathcal{B} \otimes \mathbb{S}$. By taking the product of actions F and $\varkappa_{\mathbb{S}}$, we obtain the map $F_{\mathcal{S}}\colon \mathcal{S} \to \mathcal{S} \otimes \mathcal{A}$. There exists a natural $F_{\mathcal{S}}$-invariant scalar product on $\mathcal{S}$, defined by taking the direct product of the scalar products in $\mathbb{S}$ and $\mathcal{B}$. In what follows, we shall assume that $\mathcal{S}$ is equipped with this scalar product. Let $\mathcal{S}_M \subseteq \mathcal{S}$ be the subspace of $F_{\mathcal{S}}$-invariant elements. This space is a $\mathcal{V}$-bimodule, in a natural way. In accordance with our general discussion, it is interpretable as the appropriate associated spinor bundle.

Definition 3.2. The *-algebra $\mathfrak{cl}[M]$ is called a quantum Clifford bundle algebra over the space M. The $\mathcal{V}$-bimodule $\mathcal{S}_M$ is called a quantum spinor bundle, and its elements are called quantum spinor fields over M.

It is possible to introduce a natural action map β: $\mathfrak{cl}[P] \otimes \mathcal{S} \to \mathcal{S}$ of $\mathfrak{cl}[P]$ on $\mathcal{S}$,

$$(q \otimes x)(b \otimes \zeta) = \sum_k qb_k \otimes (x \circ c_k)[\zeta]$$

This defines a faithful unital action of $\mathfrak{cl}[P]$ on $\mathcal{S}$, intertwining the corresponding natural coactions $F_{\text{cliff}} \times F_{\mathcal{S}}$ and $F_{\mathcal{S}}$, in particular $\beta(\mathfrak{cl}[M] \otimes \mathcal{S}_M) = \mathcal{S}_M$.

Proposition 3.3. The action β: $\mathfrak{cl}[P] \otimes \mathcal{S} \to \mathcal{S}$ is Hermitian for each $\psi, \varphi \in \mathcal{S}$ and $T \in \mathfrak{cl}[P]$, $\langle \psi, T\varphi \rangle = \langle T^*\psi, \varphi \rangle$.

4. QUANTUM DIRAC OPERATOR

The quantum Dirac operator acts in the quantum spinor bundle. We refer to Oziewicz, (1998) for a diagrammatic braided-algebraic foundation of the Dirac operator.

Consider a linear operator $\mathbb{D}$: $\mathcal{S} \to \mathcal{S}$ given by

$$\mathbb{D}(b \otimes x) = -i \sum_j \partial_j(b) \otimes \theta_j[x]$$

where we have interpreted the elements of $\mathbb{V}$ as linear operators in $\mathbb{S}$ in accordance with our definition of quantum Clifford algebras.

Proposition 4.1. (i) The map $\mathbb{D}$ is $F_{\mathcal{S}}$-covariant, $F_{\mathcal{S}}\mathbb{D} = (\mathbb{D} \otimes \text{id})F_{\mathcal{S}}$, $\mathbb{D}(\mathcal{S}_M) \subseteq \mathcal{S}_M$. (ii) $\langle \psi, \mathbb{D}(\varphi) \rangle = \langle \mathbb{D}(\psi), \varphi \rangle$ for each $\psi, \varphi \in \mathcal{S}$.

Definition 4.2. The map $\mathbb{D}$: $\mathcal{S}_M \to \mathcal{S}_M$ is called a quantum Dirac operator for M.

The following proposition shows that the Dirac operator contains the whole information about the differential ${}^M d$: $\Omega_M \to \Omega_M$, as in the classical theory. This fits into the axiomatic framework of Connes (1994). In contrast to Connes (1994), the eigenvalues of our Dirac operator do not obey the classical-type asymptotics in general.

Proposition 4.3. Let f and ${}^M d(f)$ be the sections of the Clifford algebra bundle. Then $[\mathbb{D}, f] = {}^M d(f)$, $\forall f \in \mathcal{V}$.

Definition 4.4. Let $g_{if} = g(\theta_i, \theta_j)$: $\mathbb{S} \to \mathbb{S}$. $\mathcal{A}$ map $\Delta_{\mathcal{S}}$: $\mathcal{S}_M \to \mathcal{S}_M$, $\Delta_{\mathcal{S}} = -\Sigma_{ij}\, \partial_i \partial_j \otimes g_{ij}$ is called the spinorial Laplacian.

A quantum generalization of the Lichnerowicz (1963) formula is as follows:

Proposition 4.5. $\mathbb{D}^2 = \Delta_{\mathcal{S}} + \hat{R}$, where $\hat{R}: \mathcal{S}_M \to \mathcal{S}_M$ is the Cliffordization of the curvature, i.e., $\hat{R} \leftrightarrow \nabla^2$ in terms of the identification $\mathrm{cl}[\mathbb{V}, g, \sigma, \Sigma] \leftrightarrow \mathbb{V}_{\hat{\Sigma}}$.

5. EXAMPLE: QUANTUM HOPF FIBRATION

We shall sketch the construction of the basic objects of the game in the case of the quantum 2-sphere (Podleś, 1987) equipped with a canonical spin structure coming from the quantum Hopf fibering. The structure group for the Hopf fibering is the classical U(1); however, the differential calculus over it will be *quantum*. It turns out that in the quantum case, the spectrum of the Dirac operator radically differs from the classical situation. This example shows that the asymptotics of the eigenvalues of the Dirac operator could be quite surprising in the noncommutative context. We are going to deal with the Dirac operator over a quantum 2-sphere. We refer to Owczarek (1999) for detailed calculations.

The quantum Hopf fibration is a quantum U(1) bundle over a quantum 2-sphere (Podleś, 1987). The total space of the bundle is given by the quantum SU(2) group, the bundle *-algebra $\mathcal{B}$ is generated by two elements $\{\alpha, \gamma\}$, and the following relations (Woronowicz, 1987a):

$$\mu \in [-1, 1]\backslash\{0\}, \qquad \alpha\alpha^* + \mu^2\gamma\gamma^* = 1, \qquad \alpha^*\alpha + \gamma^*\gamma = 1$$

$$\alpha\gamma = \mu\gamma\alpha, \qquad \alpha\gamma^* = \mu\gamma^*\alpha, \qquad \gamma\gamma^* = \gamma^*\gamma$$

The Hopf *-algebra $\mathcal{A}$ of the structure group $G = \mathrm{U}(1)$ is generated by a single unitary element U. The coproduct is $\phi(U) = U \otimes U$. The following matrix defines the fundamental representation of the quantum SU(2) group:

$$u = \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

The above relations defining $\mathcal{B}$ are equivalent to the unitarity property $u^{-1} = u^*$. The coproduct map $\phi: \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ is uniquely determined by $\phi(u_{ij}) = \Sigma_k\, u_{ik} \otimes u_{kj}$. We have used the same symbol for the coproducts on G and P.

Our structure group G is a *subgroup* of P in accordance with the identification $\mathcal{A} \leftrightarrow \mathcal{B}/\mathrm{gen}(\gamma, \gamma^*)$. If $\mu \neq 1, -1$, then G is exactly the classical part of P. The action $F: \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ is constructed from the coproduct by taking the factor projection on the second tensoriand, in other words, $F = (\mathrm{id} \otimes [\ ]_{\mathcal{A}})\phi$. The algebra $\mathcal{V}$ describing the quantum 2-sphere M is defined as the F-fixed point subalgebra of $\mathcal{B}$, so that the map i is just the inclusion.

Let us now sketch the construction of a canonical frame structure on the quantum 2-sphere (Đurđevich, 1998). The canonical 3-dimensional left-covariant and *-covariant calculus Φ over P is constructed in Woronowicz (1987a). The space Φ_{inv} is spanned by the elements

$$\eta_3 = \pi(\alpha - \alpha^*), \qquad \eta_+ = \pi(\gamma), \qquad \eta_- = \pi(\gamma^*)$$

and the canonical right $\mathcal{B}$-module structure $\circ$ on Φ_{inv} is given by

$$\mu^2\, \eta_3 \circ \alpha = \eta_3, \qquad \eta_3 \circ \alpha^* = \mu^2\, \eta_3$$

$$\Phi_{inv} \circ \gamma = \Phi_{inv} \circ \gamma^* = \{0\}$$

$$\mu\eta_\pm \circ \alpha = \eta_\pm, \qquad \eta_\pm \circ \alpha^* = \mu\eta_\pm$$

This $\mathcal{B}$-module structure on Φ_{inv} factorizes through the ideal gen $\{\gamma, \gamma^*\}$ and induces a right $\mathcal{A}$-module structure on the same space (and will be denoted by the same symbol). Let $\mathbb{V}$ be a vector space spanned by $\eta_\pm$. It will be equipped with the constructed $\circ$ and *-structures, and we shall assume that

$$\chi(\eta_+) = \eta_+ \otimes U^2, \qquad \chi(\eta_-) = \eta_- \otimes U^{-2}$$

Such a definition allows us to interpret $\chi\colon \mathbb{V} \to \mathbb{V} \otimes \mathcal{A}$ as *the adjoint* action of G coming from the group structure in P. It follows that (in the basis $\eta_\pm$) the associated braid operator $\tau\colon \mathbb{V}^{\otimes 2} \to \mathbb{V}^{\otimes 2}$ looks like

$$\tau = \begin{pmatrix} 1/\mu^2 & 0 & 0 & 0 \\ 0 & 0 & 1/\mu^2 & 0 \\ 0 & \mu^2 & 0 & 0 \\ 0 & 0 & 0 & \mu^2 \end{pmatrix}$$

The τ-exterior algebra is given by the relations

$$\eta_\pm^2 = 0, \qquad \eta_+\, \eta_- = -\mu^2\, \eta_-\, \eta_+$$

These relations are a subset of the relations defining the canonical higher order calculus over P given by the universal differential envelope $\Phi^\wedge$ of Φ (Woronowicz, 1987b; Đurđevich, 1996a, Appendix B). The remaining relations involving η_3 are

$$\eta_3^2 = 0, \qquad \eta_3\eta_\pm = \mu^{\mp 4}\eta_\pm\eta_3$$

This means that $\mathfrak{hor}_P$ is viewable as a subalgebra of $\Phi^\wedge$ generated by $\mathcal{B} = \mathfrak{hor}_P^0$ and the elements η_+, η_-. We can introduce a natural projection homomorphism $p_{\mathfrak{hor}}\colon \Phi^\wedge \to \mathfrak{hor}_P$,

$$p_{\mathfrak{hor}}(\eta_3) = 0, \qquad p_{\mathfrak{hor}}|\mathfrak{hor}_P = \mathrm{id}$$

The canonical antiderivation $\nabla: \mathfrak{hor}_P \to \mathfrak{hor}_P$ is defined as the composition of this projection with the differential $d: \Phi^\wedge \to \Phi^\wedge$.

The constructed map coincides with the covariant derivative of the canonical regular connection introduced in Đurđevich (1997). It corresponds to the standard Levi-Civita connection on the 2-sphere.

We are in the context of the spin bundles. The analog of the orthonormal frame bundle P_0 over the 2-sphere M is given by the *-subalgebra $\mathcal{B}_+ \subseteq \mathcal{B}$ corresponding to the quantum SO(3) group (even combinations of generators α, γ, γ^* α^*). The structure group $G = \mathrm{U}(1)$ is here understood as a two-fold covering of the structure group $G_0 = \mathrm{SO}(2)$ of P_0.

A braid operator σ is given by the matrix

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1/\mu^2 & 0 \\ 0 & \mu^2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the *-structure on $\mathbb{V}$ is specified by $\eta_+^* = \mu\eta_-$, while $\eta_3^* = -\eta_3$. The quantum metric is defined by

$$g_{+-}^* = g_{+-} = \mu^2 g_{-+}, \qquad g_{++} = g_{--} = 0$$

The algebra Σ is infinite dimensional. A realization of Σ in the Hilbert space $H = l^2(\mathbb{Z})$ is given by

$$g_{+-}: \quad e_k \mapsto \frac{1}{2}\,\mu^{2k}\, e_k$$

where $\{e_k | k \in \mathbb{Z}\}$ are canonical basis vectors in H. A common domain H_0 for all the operators from Σ consists of rapidly decaying sequences.

For the spinor space, we shall take $\mathbb{S} = \mathbb{C}^2$ with the canonical basis $|+\rangle$, $|-\rangle$ and the action

$$\varkappa_{\mathbb{S}}|+\rangle = |+\rangle \otimes U, \qquad \varkappa_{\mathbb{S}}|-\rangle = |-\rangle \otimes U^{-1}$$

The metric g and the algebra Σ are completely determined by the assignment

$$\eta_+ \mapsto \mu^{1/2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \eta_- \mapsto \mu^{-1/2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\frac{1}{\mu}\,\gamma[g(\eta_+, \eta_-)] = \mu\gamma[g(\eta_-, \eta_+)] = \frac{1}{2}\begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix}, \qquad \gamma[\Sigma] = \mathbb{C} \otimes \mathbb{C}$$

and the spinor representation $\gamma: \Sigma \to \mathcal{B}(\mathbb{S})$ is not faithful.

We are now going to study the Dirac operator. For each spin level $s \in \mathbb{N}/2$, let u^s be the matrix of the canonical spin-s irreducible representation of P. The matrix elements of all possible representations u^s form a natural basis in $\mathcal{B}$. Let us denote by $\mathcal{B}_s$ the subspace of $\mathcal{B}$ spanned by the matrix elements of u^s. We have

$$\mathcal{B} = \sum_{s \geq 0}^{\otimes} \mathcal{B}_s$$

The coordinate vector fields ∂_-, ∂_+ of the frame structure coincide with the spin creation and annihilation operators $iK_\mp$ for the right regular representation given by the coproduct $\phi\colon \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$. Obviously $\phi(\mathcal{B}_s) \subseteq \mathcal{B}_s \otimes \mathcal{B}_s$, and, moreover, the space $\mathcal{B}_s$ is characterized as the multiple irreducible spin-s subspace of $\mathcal{B}$. In terms of the matrix elements u^s_{ij}, the operators $\partial_\pm$ act nontrivially only on the second indexes, while the first indexes are 'free'.

Therefore, we can write

$$\mathscr{B} \leftrightarrow \underbrace{H_s \otimes \cdots \otimes H_s}_{2_s+1}$$

and introduce a canonical basis in $\mathcal{B}_s$ of the form $\{\psi^m_{\alpha s} | m, \alpha = -s, \ldots, s\}$. Here α is interpreted as a 'degeneration index'. In summary, we have

$$K_+(\psi^m_{\alpha s}) = -i\partial_+(\psi^m_{\alpha s}) = \frac{1}{\mu^{s+m}}\{s - m)_\mu(s + m + 1)_\mu\}^{1/2}\psi^{m+1}_{\alpha s}$$

$$K_-(\psi^m_{\alpha s}) = -i\partial_-(\psi^m_{\alpha s}) = \frac{\mu}{\mu^{s+m}}\{(s - m + 1)_\mu(s + m)_\mu\}^{1/2}\psi^{m-1}_{\alpha s}$$

$$F(\psi^m_{\alpha s}) = \psi^m_{\alpha s} \otimes U^m, \qquad n_\mu = \frac{1 - \mu^{2n}}{1 - \mu^2}$$

Hence the spinor module $\mathcal{S}$ is decomposed as follows:

$$\mathcal{S} = \sum_{s \in \mathbb{N} - 1/2}^{\oplus} \mathcal{S}_s$$

where the spaces $\mathcal{S}_s$ are spanned by vectors $\psi^{1/2}_{\alpha s} \otimes |-\rangle$ and $\psi^{-1/2}_{\alpha s} \otimes |+\rangle$ with the degeneracy index α arbitrary.

The Dirac operator is

$$i\mathbb{D} = \partial_+ \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \partial_- \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and it follows that $\mathbb{D}(\mathcal{S}_s) \subseteq \mathcal{S}_s$ for each $s \in \mathbb{N} - 1/2$. It is now very easy to diagonalize the reduced operators. We have two eigenvalues with the eigenvectors of the form $\mathbb{D}\psi^{+}_{\alpha s} = \lambda_s \psi^{+}_{\alpha s}$ and $\mathbb{D}\psi^{-}_{\alpha s} = - \lambda_s \psi^{-}_{\alpha s}$, where

$$\lambda_s = \frac{\mu^{s+1/2} - \mu^{-s-1/2}}{\mu - \mu^{-1}} \tag{17}$$

In the above formulas, we assumed that $\mu \geq 0$. The spectrum is invariant if we replace $\mu \mapsto -\mu$. On the other hand, if $\mu = 1, -1$, the spectrum will be given by $\lambda_s = 2s$. The case $\mu = -1$ is very special, as it gives us a quantum spin bundle over the classical 2-sphere M. It illustrates a general phenomenon that the classification problem of quantum principal bundles is qualitatively different from its classical counterpart, even if both the base manifold and the structure group are classical.

6. DISCUSSION OF QUANTUM PHENOMENA

In classical geometry, very important geometrical information is contained in the eigenvalues of elliptic operators. In particular, the distribution of eigenvalues of the Dirac operator reflects the structure of compact Riemannian spin manifolds.

If we look at the assymptotics of the spectrum of (the modulus of) the Dirac operator over the quantum 2-sphere (with their degeneracies taken into account), we arrive at the expression

$$a_N \sim \begin{cases} \sqrt{N} & \text{for} \quad \mu = \pm 1 \\ \mu^{-\sqrt{N/2}} & \text{if} \quad \mu \in (0, 1) \end{cases}$$

with the symmetry $\mu \mapsto -\mu$. In particular, we see that the inverse of the Dirac operator is *trace class* in the fully quantum case $\mu \neq 1, -1$. This is not compatible with the formulation proposed in Connes (1994), where it was assumed that the quantum Dirac operator will always have a similar asymptotics as in classical geometry

$$a_N \sim N^{1/\mathrm{d}}, \qquad d = \dim(M)$$

Quantum geometry gives us more freedom, and it is not possible to cover the diversity of all possible quantum spaces by a single asymptotic expression. In our theory (as far as we consider the pure Riemannian geometry), the group G plays the role of special orthogonal structure group $SO(d)$ and $\varkappa$ plays the role of its fundamental representation. In various interesting examples, the representation $\varkappa: \mathbb{V} \rightarrow \mathbb{V} \otimes \mathscr{A}$ will be irreducible. However, in general, this representation will be reducible. The fact that $\Sigma \neq \mathbb{C}$ allows us to overcome inherent obstacles that would appear in the formalism in the case of an irreducible $\varkappa: \mathbb{V} \rightarrow \mathbb{V} \otimes \mathscr{A}$.

This is due to the fact that the braid operators $\tau, \sigma\colon \mathbb{V} \otimes \mathbb{V} \to \mathbb{V} \otimes \mathbb{V}$, the *-structure, and the quadratic form on V are all $\varkappa$-covariant. If the metric g would take values from $\mathbb{C}$, then using elementary algebraic operations with g, * and σ, τ, we would be able to build various intertertwiners of $\varkappa$ out of these objects. In general, these intertwiners would be nonscalar operators, which implies that $\varkappa$ will be reducible. The nontriviality of Σ overcomes this obstacle.

Another quantum phenomenon is that the grade m of the volume form w is not necessarily the same as the number d of coordinate one-forms.

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